Ahmed Badr lec: 10

Higher order D.E

2nd order DE

$$y'' + P(x)y' + q(x)y = g(x)$$

i.e $\quad y'' + x^2 y' + xy = e^x$

- → Homogenous $[g(x)=0]$ → $y'' + P(x)y' + q(x)y = 0$
- → non-Homogenous $[g(x) \neq 0]$

General solution

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

y_1 & y_2 ≡ independent

- → [1] $\frac{y_2}{y_1} \neq$ const → independent

  ex: $y_1 = e^{3x}$, $y_2 = 5e^{3x}$ → $\frac{y_2}{y_1} = \frac{5e^{3x}}{e^{3x}} = 5$ → Dependent

- → [2] Wronskiam Independence Test

* y_1 , y_2

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \begin{cases} \neq 0 & \text{Independent} \\ = 0 & \text{Dependent} \end{cases}$$

ex: $y_1 = e^x$, $y_2 = e^{5x}$

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^{5x} \\ e^x & 5e^{5x} \end{vmatrix} = 5e^{6x} - e^{6x} = 4e^{6x} \neq 0$$

Independent

* y_1, y_2, y_3

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \begin{cases} \neq 0 & \text{Independent} \\ = 0 & \text{Dependent} \end{cases}$$

2nd order DE with Constant Coefficients

* Homogeneous D.E

$ay'' + by' + cy = 0$ $\quad a, b, c$ Constant

General solution

$y = C_1 y_1 + C_2 y_2$

$y = e^{mx}$, $y' = m e^{mx}$, $y'' = m^2 e^{mx}$

$\to am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$

$e^{mx}(am^2 + bm + c) = 0$ $\quad e^{mx} \neq 0$ عمرها ماتساوي صفر

Hint

$ay' + by = 0$

$\to y' = \frac{b}{a} y$

$\to y' = Ky$

$\to y = e^{mx}$

$\therefore am^2 + bm + c = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\to$ if $\sqrt{b^2-4ac} > 0 \Rightarrow m_1, m_2 \Rightarrow y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$\to$ if $\sqrt{b^2-4ac} = 0 \Rightarrow m_1 = m_2 = m \Rightarrow y = C_1 e^{mx} + C_2 x e^{mx}$

$\to$ if $\sqrt{b^2-4ac} < 0 \Rightarrow m = \alpha \mp ib \Rightarrow y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

Ex:

$\boxed{1}$ $12y'' - 5y' - 2y = 0$

let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$12m^2 - 5m - 2 = 0 \Longrightarrow m = \dfrac{5 \pm \sqrt{(5)^2 + 4(2)(12)}}{2 * 12} = \dfrac{5 \pm 11}{24}$

$\therefore m_1 = \dfrac{16}{24} = \dfrac{2}{3}$ & $m_2 = -\dfrac{1}{4} \Longrightarrow$ $y = c_1 e^{2x} + c_2 e^{-\frac{1}{4}x}$

$\boxed{2}$ $3y'' + 2y' + y = 0$

let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$3m^2 + 2m + 1 = 0 \Longrightarrow m = \dfrac{-2 \pm \sqrt{(2)^2 - 4(1)(3)}}{2 * 3} = \dfrac{-1}{3} \pm \dfrac{\sqrt{2}}{3} i$

$\alpha = \dfrac{-1}{3}$, $\beta = \dfrac{\sqrt{2}}{3} \Longrightarrow$ $y = e^{-x/3}\left(c_1 \cos \dfrac{\sqrt{2}}{3} x + c_2 \sin \dfrac{\sqrt{2}}{3} x\right)$

$\boxed{3}$ $y'' + 8y' + 16y = 0$

let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$m^2 + 8m + 16 = 0 \Longrightarrow m = \dfrac{-8 \pm \sqrt{(8)^2 - 4(16)(1)}}{2 * 1} = -4$

$m_1 = m_2 = m = -4 \Longrightarrow$ $y = c_1 e^{-4x} + c_2 x e^{-4x}$

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* non_ Homogenous D.E

$$ay'' + by' + cy = g(x)$$

$$* \quad y = y_c + y_p \quad *$$

Complementary function — Particular solution

$\boxed{1}$ Find y_c

$$\longrightarrow ay'' + by' + cy = 0$$

$$y_c = C_1 y_1 + C_2 y_2$$

$\boxed{2}$ Find y_p (Method of undetermined Coefficients)

تستخدم عند وجود :-

$\boxed{1}$ Polynomial (special = Const) [ex: x^2, $10x$, 8, ----]

$\boxed{2}$ Exponantial function ($e^{\alpha x}$) [ex: e^{5x}, e^{3x}, ----]

$\boxed{3}$ Sine or Cosine function [ex: $\sin 4x$, $\cos x$, ---]

Ex:

$$y'' - 3y' - 4y = 4x^2$$

$\boxed{1}$ Find $y_c \Longrightarrow y'' - 3y' - 4y = 0$

let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$\therefore m^2 - 3m - 4 = 0 \longrightarrow m_1 = 4,\ m_2 = -1 \longrightarrow y_c = C_1 e^{4x} + C_2 e^{-x}$

$\boxed{2}$ Find $y_p \Longrightarrow$ let $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B \quad , \quad y_p'' = 2A$

$y_p'' - 3y_p' - 4y_p = 2A - 3(2Ax + B) - 4(Ax^2 + Bx + C) = 4x^2$

$\therefore 2A - 6Ax - 3B - 4Ax^2 - 4Bx - 4C = 4x^2$

$(-4A)x^2 + (-4B - 6A)x + (2A - 3B - 4C) = 4x^2$

$\Longrightarrow -4A = 4 \longrightarrow A = -1$

$\Longrightarrow -4B - 6A = 0 \longrightarrow B = \dfrac{6A}{-4} \longrightarrow B = \dfrac{3}{2}$

$\Longrightarrow 2A - 3B - 4C = 0 \longrightarrow C = \dfrac{3B - 2A}{-4} \longrightarrow C = \dfrac{-13}{8}$

$\therefore y_p = -x^2 + \dfrac{3}{2}x - \dfrac{13}{8}$

$y = y_c + y_p \longrightarrow y = C_1 e^{4x} + C_2 e^{-x} - x^2 + \dfrac{3}{2}x - \dfrac{13}{8}$

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